Methods 5 2008

1 (a) $y'' - 4y' + 4y = e^{2x} \ln x \qquad (*)$

$m^2 - 4m + 4 = 0$

$\cancel{m = 2 \pm x} \quad (m-2)^2 = 0 \quad \Rightarrow \quad y = Ae^{2x} + Bxe^{2x}$

Let $y = A(x)e^{2x} + B(x)xe^{2x}$

$y' = A'e^{2x} + 2Ae^{2x} + B'xe^{2x} + B\cancel{x}e^{2x} + 2Bxe^{2x}$

Let $A'e^{2x} + B'xe^{2x} = 0$

$\Rightarrow \quad y' = 2Ae^{2x} + Be^{2x} + 2Bxe^{2x}$

$\Rightarrow \quad y'' = 2A'e^{2x} + 4Ae^{2x} + B'e^{2x} + 2Be^{2x}$
$\qquad + 2B'xe^{2x} + 2Be^{2x} + 4Bxe^{2x}$

$(*): \quad 2A'e^{2x} + \cancel{4Ae^{2x}} + B'e^{2x} + \cancel{2Be^{2x}}$
$\qquad + 2B'xe^{2x} + \cancel{2Be^{2x}} + \cancel{4Bxe^{2x}}$
$\qquad - \cancel{8Ae^{2x}} - \cancel{4Be^{2x}} - \cancel{8Bxe^{2x}}$
$\qquad + \cancel{4Ae^{2x}} + \cancel{4Bxe^{2x}} = \cancel{0}\, e^{2x} \ln x$

$\Rightarrow \quad 2A'e^{2x} + B'e^{2x} + 2B'xe^{2x} = \cancel{0}\, e^{2x} \ln x$

and $\quad A'e^{2x} + B'xe^{2x} = 0$

$\cancel{\Rightarrow B' = 0}$
$\cancel{\Rightarrow A' = 0}.$ $\quad \Rightarrow \quad B' = \ln x \quad \Rightarrow \quad B = \cancel{\ }\ x\ln x - x$

and $A' = \cancel{\ } -x \ln x$

$\cancel{A = -x(x\ln x - x) + \int x \ln x} - x$ $\quad ((-A$

$\cancel{2A = -x^2 \ln x - x^2 - x}$

~~ch~~

$A' = -x \ln x$

$$A = \ln x\left(-\frac{x^2}{2}\right) + \int \frac{1}{x}\frac{x^2}{2}\,dx$$

$$= \ln x\left(-\frac{x^2}{2}\right) + \frac{1}{2}\int x\,dx$$

$$= \ln x\left(-\frac{x^2}{2}\right) + \frac{x^2}{4}$$

$$\Rightarrow \quad y = \left[-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right]e^{2x} + (x\ln x - x)\,x e^{2x}$$

(b) $y = \int e^{xt} f(t)\,dt$.

$$\int \left[xt^2 - (4x-1)t + (4x-2)\right] f(t)\, e^{xt}\,dt$$

If this is $\int \frac{d}{dt}\left[g e^{xt}\right] dt$

$$= \int \left(g' e^{xt} + x g e^{xt}\right) dt$$

$$\Rightarrow \quad g = (t^2 - 4t + 4) f$$

$$g' = (t-2) f$$

$$\Rightarrow \quad \frac{g'}{g} = \frac{t-2}{(t-2)^2} = \frac{1}{t-2}$$

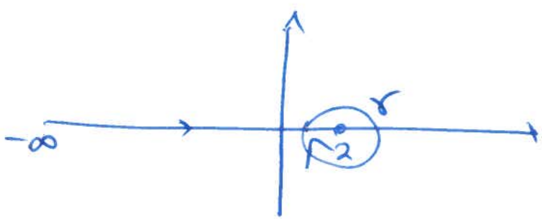
$$\Rightarrow \quad g = t-2 \quad \Rightarrow \quad f = \frac{1}{t-2}$$

$$\Rightarrow \quad y = \int e^{xt}\frac{1}{t-2}\,dt.$$

Pick path s.t. $y \neq 0$ and $[e^{xt} g]_\gamma = 0$

$[e^{xt}(t-2)]_\gamma = 0$

$t = 2, -\infty$



so one is $y = \int_{-\infty}^{2} e^{xt} \frac{1}{t-2} \, dt$

other is $y = \oint_\gamma e^{xt} \frac{1}{t-2} \, dt$

$= 2\pi i \, \mathrm{Res}(t=2)$

$\underline{2\pi i \, (e^{2x})}$.

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2. $\ddot{x} + f(\dot{x}) + g(x) = 0$ $\quad y = \dot{x}$

$\rightarrow \begin{cases} \dot{y} = -f - g \\ \dot{x} = y \end{cases}$

(a) Periodic solutions have form $\left.\begin{array}{l} y(t_0+T) = y(t_0) \\ x(t_0+T) = y(t_0) \end{array}\right\}$ $\forall\, t_0$ for some T

these will be closed trajectories.

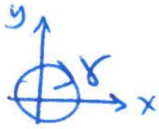
(b) $\ddot{x} + f(\dot{x}) + g(x) = 0$

$\Leftrightarrow \dot{y} + f(y) + g(x) = 0$

$\Leftrightarrow \frac{dy}{dx} \dot{x} + f(y) + g(x) = 0$

$\frac{dy}{dx} y + f(y) + g(x) = 0.$

Assume closed orbit exists, γ. Integrate around.



$$\int y \frac{dy}{dx}\, dx + \int f(y)\, dx + \int g(x)\, dx = 0$$

$$\underbrace{\left[\tfrac{1}{2}y^2\right]_\gamma}_{=0} + \int f(y)\, dx + \underbrace{[G]_\gamma}_{=0} = 0$$

$$\Rightarrow \int f(y)\, dx = 0$$

$$\Rightarrow \int f(y) \frac{dx}{dt}\, dt = 0$$

$$\Rightarrow \int f(\dot{x})\dot{x}\, dt = 0$$

$\Rightarrow$ can't have closed orbit if $\dot{x} f(\dot{x})$ has single sign.

(c) $\ddot{x} + \dot{x} - x^2 + x = 0$

$\dot{y} + y - x^2 + x = 0$

$\dot{y} = -y + x^2 - x = Q$ $\qquad \dot{x} = y$ "P"

Sings when $\dot{y} = \dot{x} = 0$ ie $y = 0,\ x = 0, 1$

$\dot{y} = 0$

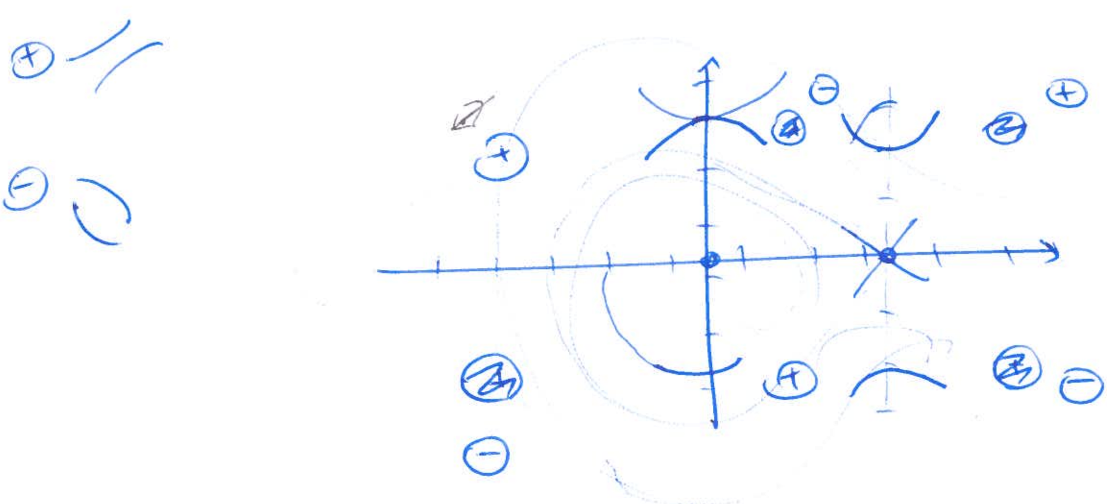
$y = x(x-1)$

$\frac{dy}{dx} = \frac{-y + x^2 - x}{y}$

at $-1, 1$

$\frac{dy}{dx} = \frac{-1+1+1}{1}$

(+ve)



$$J = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2x-1 & -1 \end{pmatrix}$$

$$J\big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} . \quad \text{evals} \quad \begin{vmatrix} \lambda & -1 \\ 1 & \lambda+1 \end{vmatrix} = 0$$

$$\lambda(\lambda+1)+1 = 0$$
$$\lambda^2+\lambda+1 = 0$$
$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2}$$

complex
neg. real part $\Rightarrow$ spiral
stable.

$$J\big|_{\cancel{(2,0)}\ \cancel{(0,1)}\ (1,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{evals} \quad \begin{vmatrix} \lambda & -1 \\ -1 & \lambda+1 \end{vmatrix} = 0$$

$$\lambda(\lambda+1)-1 = 0$$
$$\lambda^2+\lambda-1 = 0$$
$$\lambda = \frac{-1 \pm \sqrt{1+4}}{2}$$

real different sign
saddle.

Separatrices at 1.0.

locally $\quad \dfrac{dY}{dX} = \dfrac{X - Y}{Y}$

$$Y = mX \Rightarrow m = \frac{X-mX}{mX} = \frac{1-m}{m} \quad \cancel{\frac{1}{m}-1}$$

$$\Rightarrow m^2 = 1-m \Rightarrow m^2+m-1 = 0$$
$$m = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

3. $\frac{d}{dt} \to n\frac{d}{d\theta}$

$$\Rightarrow \quad n^2 x'' + x - \varepsilon n (x')^3 + \varepsilon n x' = 0$$

$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2$

$n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2$

$$[n_0 + \varepsilon n_1 + \varepsilon^2 n_2][x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'']$$

$$+ [x_0 + \varepsilon x_1 + \varepsilon^2 x_2] - \varepsilon[n_0 + \varepsilon n_1 \cancel{+\varepsilon^2 n_2}][x_0'^3 + 3x_0'^2 x_1' \varepsilon]$$

$$+ \varepsilon[n_0 + \varepsilon n_1][x_0' + \varepsilon x_1'] \quad \cancel{+ 3 x_0'^2 \varepsilon^2 x_2 + 3\varepsilon^2 x_1'^2 x_0}$$

$[\varepsilon^0]$ $n_0 x_0'' + x_0 = 0$

$$\Rightarrow \quad x_0 = A\sin\left(\frac{t}{n_0}\right) + B\cos\left(\frac{t}{n_0}\right)$$

$x(0) = 0 \Rightarrow B = 0$

$x(t) = x(t + 2\pi) \Rightarrow \underline{n_0 = 1}$.

$\underline{x_0 = A\sin t}$.

$[\varepsilon^1]$ $n_1 x_0'' + x_1'' + x_1 - x_0'^3 + x_0' = 0$

$$x_1'' + x_1 = -n_1 x_0'' + x_0'^3 - x_0'$$

$$= +n_1 A\sin t + A^3\cos^3 t - A\cos t$$

$$= n_1 A\sin t + A^3 \tfrac{1}{4}\cos 3t + A^3 \tfrac{3}{4}\cos t - A\cos t$$

$[\cos^3 t = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta]$

To remove sin & cos terms from RHS,

$$n_1 = 0$$

$$A^3 \frac{3}{4} = A$$

$$A^2 = \frac{4}{3} \qquad A = \frac{2}{\sqrt{3}} . \qquad A^3 = 8\left(\frac{1}{3}\right)^{3/2}$$

$$\Rightarrow \quad x_1'' + x_1 = A^3 \frac{1}{4} \cos 3t$$

$$= 2\left(\frac{1}{3}\right)^{3/2} \cos 3t$$

$$\Rightarrow \quad x_1 = C\cos t + D\sin t + E2\left(\frac{1}{3}\right)^{3/2} \cos 3t$$

$$x(0) = 0 \Rightarrow \quad C = 0$$

$$x_1 = D\sin t + E2\left(\frac{1}{3}\right)^{3/2} \cos 3t$$

$$x_1'' = -D\sin t - 9E\, 2\left(\frac{1}{3}\right)^{3/2} \cos 3t$$

$$\Rightarrow \quad -8E\, 2\left(\frac{1}{3}\right)^{3/2} \cos 3t = 2\left(\frac{1}{3}\right)^{3/2} \cos 3t$$

$$\Rightarrow E = -\frac{1}{8} .$$

$$\Rightarrow \quad \underline{x_1 = \frac{1}{4}\left(\frac{1}{3}\right)^{3/2} (\cos\theta - \cos 3\theta) + B_1 \sin\theta}$$

and $\underline{n = 1 + \varepsilon^2 n_2}$.

$[\varepsilon^2]$ $\cancel{A_2 X_0'' + A_1 X_1''} + A_0$ etc.

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4. $\ddot{x} + \varepsilon f(x, \dot{x}) + x = 0$

Let $\cancel{x=}$ $\cancel{t=}$ $T = \varepsilon t$

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + \frac{\partial}{\partial T}\frac{\partial T}{\partial t} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}$$

$$x = x_0 + \varepsilon x_1$$

$$\therefore \left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}\right)^2 (x_0 + \varepsilon x_1) + \varepsilon f + (x_0 + \varepsilon x_1) = 0$$

$[\varepsilon^0]$
$$\Rightarrow \frac{\partial^2 x_0}{\partial t^2} + x_0 = 0 \quad \Rightarrow \quad x_0 = A(T)\sin(t + \Phi(T))$$
$$= A(T)\sin\chi$$

$[\varepsilon^1]$
$$\frac{\partial^2 x_1}{\partial t^2} + \frac{\partial^2}{\partial t \partial T} x_0 + f + x_1 = 0$$

$$\frac{\partial}{\partial t} x_0 = A\cos\chi$$

$$\frac{\partial^2}{\partial t \partial T} x_0 = A'\cos\chi - A\sin\chi\,\Phi'$$

$$\Rightarrow \ddot{x}_1 + x_1 = -f - A'\cos\chi + A\sin\chi\,\Phi'.$$

$\int \cos x$:

$$0 = -\int_{-\pi}^{\pi} f \cos x \, dx \quad - A'\pi$$

$$\Rightarrow A' = \frac{1}{\pi}\int_{-\pi}^{\pi} f \cos x \, dx.$$

$\times \int \sin x$:

$$0 = -\int f \sin x \, dx + A\Phi'\pi$$

$$\Rightarrow \Phi' = \frac{1}{A\pi}\int f \sin x \, dx.$$

$$A' = \cancel{\frac{1}{\pi}\int_{-\pi}^{\pi} (4x^2-1)\frac{dx}{dt} \cos x \, dx}$$

$$f(x, \dot{x}) = (4x^2-1)\dot{x}$$

$$\Rightarrow f(A\sin x, A\cos x) = (4A^2\sin^2 x - 1)(A\cos x)$$

$$= [4A^2(1-\cos^2 x) - 1]A\cos x$$

$$= 4A^3\cos x - 4A^3\cos^3 x - A\cos x$$

$$\Rightarrow A' = \frac{1}{\pi}\int_{-\pi}^{\pi}[4A^3\cos^2 x - 4A^3\cos^4 x - A\cos^2 x]\,dx$$

$$= \frac{1}{\pi}[4A^3\pi - 4A^3\tfrac{3\pi}{4} - A\pi]$$

$$= 4A^3 - A^3 3 - A$$

$$\frac{dA}{dT} = A^3 - A = A(A^2-1)$$

$$\frac{dA}{dT} = A(A^2-1) = A(A+1)(A-1)$$

$$\int \frac{1}{A(A+1)(A-1)} = \int dT$$

$$\frac{a}{A} + \frac{b}{A+1} + \frac{c}{A-1} = a(A+1)(A-1) + bA(A-1) + c(A+1)A$$

$$= a(A^2-1) + b(A^2-A) + c(A^2+A)$$

$\Rightarrow a+b+c=0 \quad \Rightarrow \cancel{b=2a} \quad -1+b+b=0$

$-b+c=0 \quad \Rightarrow b=c$

$a=-1. \quad \Rightarrow$

$\Rightarrow b=c=\frac{1}{2}$

$a=-1.$

Hang on! periodic solution is

$$\frac{dA}{dT}=0 \Rightarrow A=1.$$

OK continue

$$-\ln A + \tfrac{1}{2}\ln(A+1) + \tfrac{1}{2}\ln(A-1) = T + \tilde{K}$$

$$\Rightarrow \frac{(A+1)^{1/2}(A-1)^{1/2}}{A} = Ce^{T}$$

$$\left(\frac{(A+1)(A-1)}{A^2}\right)^{1/2}$$

$$= \left(\frac{A^2-1}{A^2}\right)^{1/2} = \left(1-\frac{1}{A^2}\right)^{1/2}$$

$$\Rightarrow \left(1-\frac{1}{A^2}\right)^{\cancel{1/2}} = Ce^{2T}$$

$$\Rightarrow C = \left(1-\frac{1}{A_0}\right)$$

and it falls out.

limit cycle sol^n when $t\to\infty$ is just sin.

5. If $I(x) = \int_0^T e^{-xt} \, f(t)\,dt$

and $f(t) \sim t^{\lambda} \sum a_n t^{\lambda_n}$ as $t \to 0$ $\quad \lambda > -1$, $\lambda_0 = 0$

& $f(t)$ does not grow superexponentially if $T = \infty$ as $t \to \infty$

$$\Rightarrow I(x) \sim \sum_{n=0}^{\infty} \frac{a_n (\lambda + \lambda_n)!}{x^{\lambda + \lambda_n + 1}}$$

(a) $\int_0^1 t^x \ln(1 - \ln t)\,dt$

$$\int_0^1 e^{x \ln t} \ln(1 - \ln t)\,dt$$

$-u = \ln t$

$-du = \frac{1}{t}\,dt$

$dt = -t\,du = -e^{-u}\,du$

$$-\int_{\infty}^{0} e^{-xu} \ln(1+u) e^{-u}\,du$$

$$= \int_0^{\infty} e^{-xu} \underbrace{\left[\ln(1+u) e^{-u}\right]}_{f(u)}\,du.$$

$$f(u) \sim \left(u - \tfrac{1}{2}u^2\right)(1 - u)$$

$$= u - \tfrac{3}{2}u^2 = u\left(1 - \tfrac{3}{2}u\right)$$

$\lambda = 1$

$a_0 = 1 \quad a_1 = -\frac{3}{2}$

$\lambda_0 = 0 \quad \lambda_1 = 1$

$$I(x) \sim \frac{1}{x^2} - \frac{3}{2}\,\frac{2}{x^3}$$

$$= \underline{\underline{\frac{1}{x^2} - \frac{3}{x^3}}}$$

(b) $\int_{\pi/2}^{\pi} e^{-xt} \sin(t) \sinh\left(t-\frac{\pi}{2}\right) dt$

Let $\cancel{t} = u = t - \frac{\pi}{2}$ $\qquad t = u + \frac{\pi}{2}$

$dt = du$

$$e^{-\frac{x\pi}{2}} \int_0^{\pi/2} e^{-xu} \sin\left(u+\frac{\pi}{2}\right) \sinh(u)\, du$$

$$= e^{-x\pi/2} \int_0^{\pi/2} e^{-xu} \underbrace{\cos u \sinh u}_{f}\, du$$

$$f \sim \left(1-\frac{u^2}{2}\right)\left(u+\frac{u^3}{3!}\right) - u - \frac{u^3}{3}$$

$$= u\left(1-\frac{u^2}{6}\right)$$

$\lambda = 1$ $\qquad \lambda_0 = 0$

$a_0 = 1$ $\qquad \lambda_1 = 1$

$a_1 = 0$ $\qquad \lambda_2 = 2$

$a_2 = -\frac{1}{6}$

$$\Rightarrow I(x) \sim e^{-x\pi/2}\left[\frac{1}{x^2} - \frac{1}{3}\frac{6}{x^4}\right]$$

$$= e^{-x\pi/2}\left(\frac{1}{x^2} - \frac{2}{x^4}\right)$$

Rest bookwork.